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The preceding considerations lead to some direct extensions of the most fundamental rules of operating with positive and negative numbers. For example, the rule for the sums of two real numbers with opposite signs is included in the following: To obtain the sum of a number on a given ray and a number on the extended ray* find the difference of their absolute values and prefix to this the angle of that one whose absolute value is the greater. The fact that the sign of the product of a positive and a negative number is negative is included in the statement that the angle of the product of a number on the α_0 ray and a number on its extension is $2\alpha_0 + \pi$. From this it results that the necessary and sufficient condition that a number is real is that a negative number is obtained by multiplying it by a number on the extension of its ray, and the necessary and sufficient condition that a number is a pure imaginary is that a positive number is obtained when we multiply the number into any number on the extended ray.

In the light of these results the question whether the terms positive and negative, which do not exhibit any evidence of the fact that they represent two special values of the possible amplitudes of a number, should be replaced, at least in theoretic work, by terms which exhibit their places in the infinite series of amplitudes, assumes a deeper meaning. If one should be asked to defend the terms 0-numbers and π -numbers it would be merely necessary to reply that the adjectives express the values of the amplitudes of these numbers, but no such rational defense could be made for the terms positive numbers and negative numbers. At any rate this view point seems to deserve notice since the question involved is so fundamental, and the present note makes no other claims for usefulness or novelty than the presentation of a very elementary and important matter in a somewhat new light.

ON CERTAIN PROPERTIES OF THE ORBITS OF A PARTICLE SUBJECT TO A CENTRAL FORCE VARYING AS AN INTEGRAL POWER OF THE DISTANCE.†

By E. J. MOULTON and F. H. HODGE, The University of Chicago.

In this paper are discussed the orbits of a particle moving subject to a central force varying as an integral power of the distance. The discussion is made with regard to their concavity and convexity, the number and distances of their apses, the ranges of values through which their radial dis-

*If the angles of two rays differ by π each one of them is said to be the extension of the other.

†This discussion had its origin in an exercise given by Professor F. R. Moulton to his class in Celestial Mechanics. It was read before the Chicago Section of the American Mathematical Society, April 18, 1908.

tances may vary, the amplitudes of the angles described by the radii vectors, and the time required for the maximum variation of the radial distance of the particle from the center of force. The discussion will be made from the implicit relations among the variables obtained from the first integrals.

It is shown in works on mechanics that the differential equations of the motion of a particle subject to a central force are

$$(1) \quad \begin{cases} f = h^2 u^3 \left(n + \frac{d^2 u}{d\theta^2} \right), & (n \geq 0), \\ \frac{d\theta}{dt} = hu^2, \end{cases}$$

where f is the force, h is the constant of areas, u is the reciprocal of the radial distance to the particle, t is the time, and θ is the angle between the radius vector and some arbitrarily chosen reference line, the origin being at the center of force. If the force varies as an integral power of the distance we have

$$(2) \quad f = \pm m^2 h^2 u^a,$$

where $m^2 h^2$ is an arbitrary constant not equal to zero, and a is an integer. The positive sign signifies that it is an attractive force; the negative sign a repulsive force. From (1) and (2) we obtain

$$(3) \quad \begin{cases} u + \frac{d^2 u}{d\theta^2} = \pm m^2 u^{a-2} & (u \geq 0), \\ \frac{d\theta}{dt} = hu^2, \end{cases}$$

as the differential equations of motion.

Multiplying both sides of the first equation of (3) by $2\frac{du}{d\theta}$ and integrating, it is found that

$$(4) \quad \begin{cases} u^2 + \left(\frac{du}{d\theta} \right)^2 = \pm \frac{2m^2}{a-1} u^{a-1} + c, & (a \neq 1, u \geq 0), \\ u^2 + \left(\frac{du}{d\theta} \right)^2 = \pm 2m^2 \log u + c, & (a = 1, u \geq 0), \end{cases}$$

where c is a constant of integration which is determined by the initial conditions. We then have

$$(5) \quad \begin{cases} \theta = \int_{u_0}^u \frac{du}{\sqrt{\pm \frac{2m^2}{\alpha-1} u^{\alpha-1} - u^2 + c}}, & (\alpha \neq 1), \\ \theta = \int_{u_0}^u \frac{du}{\sqrt{\pm 2m^2 \log u - u^2 + c}}, & (\alpha = 1). \end{cases}$$

It is seen that when $\alpha = -1, +2, 3, 4$, or 5 , u is a trigonometric, exponential, or elliptic function of θ , and the characteristics of the orbits are readily obtained. But when α is any other integer, u is in general not expressible by means of functions of θ which give readily the characteristics of the orbits.

It will be shown that by considering equations (3) and (4) the characteristic properties of the orbits for all integral values of α can be readily determined.

It is seen from physical considerations that for attractive forces the orbits are always concave toward the origin, and for repulsive forces always convex. This is evident also from equations (3) since a curve is concave toward the origin if $u + \frac{d^2 u}{d\theta^2} > 0$, and convex if $u + \frac{d^2 u}{d\theta^2} < 0$.

It is also to be noticed that when u changes from positive to negative the left member of the first equation of (3) changes sign, and that the right member changes sign if α is odd, but retains its sign if α is even. Hence, for $+m^2$ a negative value of u is allowable for attraction if α is odd, but must be considered as indicating repulsion if α is even. For $-m^2$ a corresponding statement is true. Remembering this, we can write (4) as follows:

$$(6) \quad \begin{cases} \left(\frac{du}{d\theta}\right)^2 = \pm \frac{2m^2}{\alpha-1} u^{\alpha-1} - u^2 + c \equiv \phi(u), & (\alpha \neq 1), \\ \left(\frac{du}{d\theta}\right)^2 = \pm 2m^2 \log u - u^2 + c \equiv \phi(u), & (\alpha = 1). \end{cases}$$

We get from (3) and (6),

$$(7) \quad dt = \frac{d\theta}{h u^2} = \frac{du}{h u^2 \sqrt{\phi(u)}}.$$

Now $\frac{du}{d\theta} = 0$ is a necessary and sufficient condition for an apse since, if $\frac{du}{d\theta} = 0$, then $\frac{dr}{d\theta} = 0$, ($r = \frac{1}{u}$); that is, the orbit is perpendicular to the radius vector. And since $\left(\frac{du}{d\theta}\right)^2 \equiv \phi(u)$ the number and values of the roots of $\phi(u) = 0$ determine the number and distances of the apses of the orbits.

We can then discuss the orbits from the consideration of the graphs of $\phi(u)$. This is easily obtained by adding the corresponding ordinates of the graphs whose equations are, for $a \neq 1$,

$$y_1 = \pm \frac{2m^2}{a-1} u^{a-1}, \quad y_2 = -u^2, \quad y_3 = c;$$

and for $a=1$,

$$y_1 = \pm 2m^2 \log u, \quad y_2 = -u^2, \quad y_3 = c.$$

It appears in the construction of the graphs that the type of curve is different for different values of a , and that it is necessary to consider the following cases: I, $a > 3$ and odd; II, $a > 3$ and even; III, $a = 3$; IV, $a = 2$; V, $a = 1$; VI, $a < 1$ and even; and VII, $a < 1$ and odd. It is also found that, except for $a = 3$, we get the same types of curves, having the same number of maxima and minima, and giving the same combinations of real positive, real negative, zero, and multiple roots of $\phi(u) = 0$, by taking a special value of the parameter m^2 and varying the parameter c , as by varying both m^2 and c . There is no loss in the generality of the results if we consider only a single value of m^2 , with the exceptional case $a = 3$, where the type of curve depends upon whether m^2 is greater than, equal to, or less than, unity.

The discussion is divided into seven cases, depending upon the value of a , and these are subdivided according to the value of c . The first case is given somewhat in detail, but in the other cases only the special peculiarities are pointed out. The time required for the maximum variation of u (which is denoted by T) is discussed after the other characteristics have been ascertained. Let Θ denote the angle described by the radius vector for the maximum variation of u .

CASE I. $a > 3$ AND ODD.

The types of graph of $\phi(u)$ for the different values of c are shown in Fig. 1, the sub-cases being (a), (b), (c), (d), and (e) for attraction, and (f), (g), and (h) for repulsion. Since a is odd, both positive and negative u 's may be used.

Sub-Case (a). The characteristics of the orbits in this case are, (1) they have no apse, (2) they extend from the origin to infinity, and (3) Θ is finite.

It is evident that, since $\phi(u) = 0$ has no real roots (as is seen from the graph), there can be no apse. If the initial conditions are $\theta = 0$, $u = u_0$, $\frac{du}{d\theta} = \frac{du_0}{d\theta}$, then, since $\left(\frac{du}{d\theta}\right)^2 > 0$ for every u , $\frac{du_0}{d\theta}$ may be either positive or negative, but cannot be zero. If $\frac{du_0}{d\theta} > 0$, then as θ increases u increases

until $\frac{du}{d\theta}$ becomes negative, which can occur only by $\left(\frac{du}{d\theta}\right)$ passing through

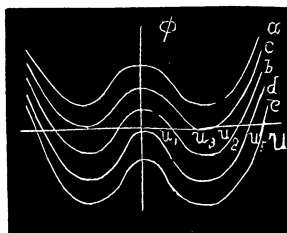


Fig. I. $a > 3$ and odd.
Attraction.

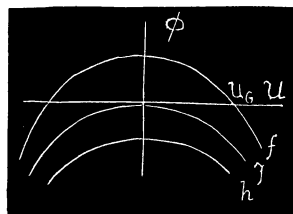


Fig. I. $a > 3$ and even.
Repulsion.

infinity. Since $\left(\frac{du}{d\theta}\right)^2$ becomes infinite only for u infinite, u increases indefinitely. If $\frac{du_0}{d\theta} < 0$, it follows from a similar argument that u decreases to negative infinity. Hence u may vary from $+\infty$ to $-\infty$. That is, $r=1/u$ may vary from 0 to $+\infty$, change to $-\infty$ and return to 0; or the orbit may extend from the origin to infinity and back to the origin.

The proof that Θ is finite comes from equations (5). Since the denominator of the integrand is never zero, and since $1/(\phi(\infty))$ is infinite of order greater than unity, $\Theta = \int_0^\infty \frac{du}{1/\phi(u)}$ is finite. This means that the orbit makes a finite number of revolutions around the origin between zero and infinity. The type of orbits for this case is shown in Fig. A.

Sub-Case (b). The characteristics of the orbits are (1) they have two apse distances, a single orbit, however, having only one apse distance, (2) the orbits lie either entirely within one circle, or entirely without a larger circle with the same center, the first orbit going to the origin, the second to infinity, and (3) Θ is finite for each orbit.

The real roots of $\phi(u)=0$ are $\pm u_1$, and $\pm u_2$; hence there are apses at $r=1/u_1$ and $r=1/u_2$. But a single orbit can have only one of these apse distances, since $\theta = \int_{u_1}^{u_2} \frac{du}{1/\phi(u)}$ is imaginary, giving an imaginary orbit, between these two apse distances. The discussion of the orbits in this case is given in three parts:

(1) If the initial conditions are $\theta=0$, $u=u_0$, $0 < |u_0| < u_1$, $\frac{du}{d\theta} = \frac{du_0}{d\theta}$; then, if $\frac{du_0}{d\theta} > 0$, u increases and, as may be seen from the graph, $\frac{du}{d\theta}$ decreases, becoming zero at $u=u_1$. Now $\frac{d^2u}{d\theta^2} = \frac{1}{2} \frac{d\phi(u)}{du}$. Therefore, as



Fig. A.

is seen from the graph, $\frac{d^2u}{d\theta^2} \neq 0$ at $u=u_1$. Hence $\frac{du}{d\theta}$ continues to decrease and changes sign at $u=u_1$, and u begins to decrease. $\frac{du}{d\theta}$ does not change sign again until $u=-u_1$, where it becomes positive and u again increases to $u=+u_1$. This varying of u between $-u_1$ and $+u_1$ continues indefinitely as θ increases. If $\frac{du_0}{d\theta} < 0$, the same range of variation of u results. We have then for the variation of r , $-u_1 \leq \frac{1}{r} \leq +u_1$ or $\frac{1}{u_1} \leq |r| \leq \infty$. That is, the orbits lie outside the circle whose equation is $r=1/u_1$.

(2) If initially, $u=u_0$, $u_1 < |u_0| < u_2$, then $\theta = \int_{u_0}^{u'} \frac{du}{\sqrt{\phi(u)}}$, ($u_1 < |u'_0| < u_2$), is imaginary. This means that there is no real orbit for these initial conditions and the given value of c .

(3) If the initial conditions are $\theta=0$, $u=u_0$, $u_2 < u_0$, $\frac{du}{d\theta} = \frac{du_0}{d\theta}$, then, if $\frac{du_0}{d\theta} > 0$, u increases indefinitely. If $\frac{du_0}{d\theta} < 0$, u decreases to u_2 where $\frac{du}{d\theta}$ becomes zero and changes sign, after which u increases indefinitely. If $u_0 < -u_2$ then $|u|$ has the same range of variation as if $u_0 > u_2$; namely, $u_2 \leq |u| \leq \infty$. Hence we have $0 \leq |r| \leq 1/u_2$; that is, the orbits lie inside the circle whose equation is $r=1/u_2$, and have apses on this circle.

Since $\sqrt{[\phi(u)]}$ is an infinitesimal of order less than unity at $u=u_1$, $\Theta = \int_0^{u_1} \frac{du}{\sqrt{\phi(u)}}$ is finite. $\sqrt{[\phi(u)]}$ is also an infinitesimal of order less than unity at $u=u_2$ and is infinite of order greater than unity at $u=\infty$; therefore $\Theta = \int_{u_2}^{\infty} \frac{du}{\sqrt{\phi(u)}}$ is finite.

The type of orbits for this case is shown in Fig. B.

Sub-Case (c). The characteristics of the orbits are (1) they have one apse distance, (2) they either extend from the origin to a certain circle whose center is at the origin, or always move on that circle, or extend from that circle to infinity, and (3), Θ is infinite for those orbits, which are not circles, the orbits approaching the circles asymptotically both from the inside and from the outside.

The real roots of $\phi(u)=0$ are $\pm u_3$; hence there is an apse at $r=1/u_3$. The discussion of the orbits is again broken up into three parts.

(1) If the initial conditions are $\theta=0$, $u=u_0$, $0 \leq |u_0| < u_3$,

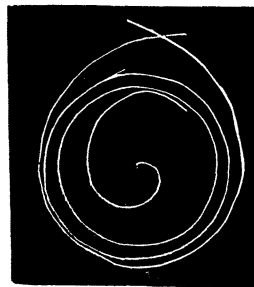


Fig. B.

$\frac{du}{d\theta} = \frac{du_0}{d\theta}$, then, if $\frac{du_0}{d\theta} > 0$, u increases to $+u_3$; and if $\frac{du_0}{d\theta} < 0$, u decreases to $-u_3$. Since $\phi(u)=0$ has a double root at $u=\pm u_3$, $\frac{1}{\sqrt{\phi(u)}}$ is infinite of the first order at $\pm u_3$. Hence $\Theta_1 = \int_{u_0}^{\pm u_3} \frac{du}{\sqrt{\phi(u)}}$ is infinite. That is, the orbit approaches the circle $r=1/u_3$ asymptotically from the outside. Since $\frac{1}{\sqrt{\phi(u)}}$ is finite except at $u=\pm u_3$, the angle described by the radius vector going along the orbit from any point outside the circle to infinity is finite.

(2) If the initial conditions are $\theta=0$, $u=u_0=\pm u_3$, then $\frac{du_0}{d\theta}=0$, and $\frac{d^2 u_0}{d\theta^2}=0$. Since $\theta = \int_{u_3}^{u'} \frac{du}{\sqrt{\phi(u)}}$ is infinite for any u' different from u_3 , the orbit is the circle $r=1/u_3$.

(3) If the initial conditions are $\theta=0$, $u=u_0$, $u_0 > u_3$, $\frac{du}{d\theta} = \frac{du_0}{d\theta}$, then if $\frac{du_0}{d\theta} > 0$, u increases indefinitely as θ increases, and $\theta_1 = \int_{u_0}^{\infty} \frac{du}{\sqrt{\phi(u)}}$ is finite. If $\frac{du_0}{d\theta} < 0$, u decreases to u_3 , and $\theta_2 = \int_{u_0}^{u_3} \frac{du}{\sqrt{\phi(u)}}$ is infinite. That is, r decreases to zero from any point inside the circle $r=1/u_3$ for a finite value of θ , and increases to $1/u_3$ for an infinite value of θ . The orbit approaches asymptotically the circle $r=1/u_3$. The same orbit is obtained for $u_0 < -u_3$.

Sub-Case (c) may be considered as the limiting case of *sub-case (b)*, as the two circles approach coincidence. From this point of view one would say that the orbits have apsides at $r=1/u_3$. It is also a limiting case of *sub-case (a)*; whence the orbits might be considered as extending from the origin to infinity.

The type of orbits for this case is shown in Fig. C.

Sub-Case (d). The characteristics of the orbits are, (1) they are either circles of infinite radius or finite orbits having one apse distance, (2) the finite orbits go to the origin, and (3) Θ is finite. The discussion in this case is essentially the same as that in *sub-case (b)*. The outer orbit has become a circle at infinity.

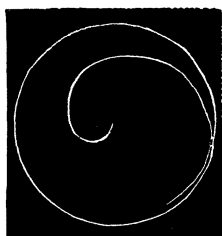


Fig. D.

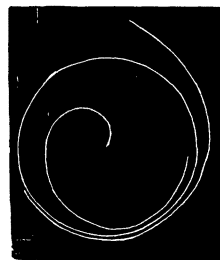


Fig. C.

Sub-Case (e). This is the same as the preceding sub-case except that the orbits at infinity have vanished.

Sub-Case (f). Here we are dealing with a repulsive force. The characteristics of the orbits are,

(1) they have one apse distance, and (2) they go to infinity, having a finite Θ . The discussion is made in the same manner as previously. It is seen that, since the orbit is convex toward the origin, $\Theta < \pi$.

Sub-Case (g). The orbit has gone to infinity.

Sub-Case (h). No real orbits exist.

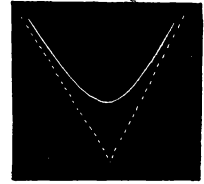


Fig. E.

CASE II. $a > 3$ AND EVEN.

The types of graph of $\phi(u)$ for the different values of c are shown in Fig. II. Since a is even, a negative value of u must be considered as indicating a repulsive force. We get all types of orbits for both attraction and repulsion if we consider only $+m^2$ in the discussion of this case. As u passes through zero, that is, as the orbit passes through infinity, the force changes from attraction to repulsion. The discussion of the sub-cases involves no new features.

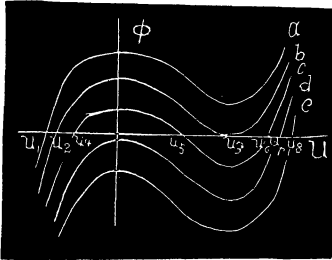


Fig. II. $a > 3$ and even.

The characteristics of the orbits in all cases are shown in the Table of Results, in which the italics indicate repulsive forces. The graphs are numbered to correspond to the cases. For $a=3$ the graph of $\phi(u)$ and the types of orbits for repulsion are the same as in Case I, and these are not repeated. In Case V the graphs of $\phi(u)$ for both attraction and repulsion are shown in the same figure; for all negative values of u , $\phi(u)$ is imaginary.

The following are peculiarities of the orbits not already discussed.

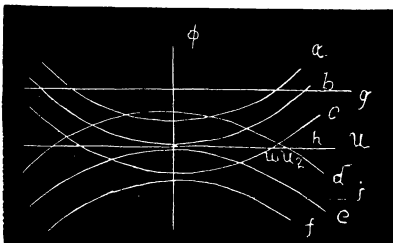


Fig. III. $a=3$.
Attraction.

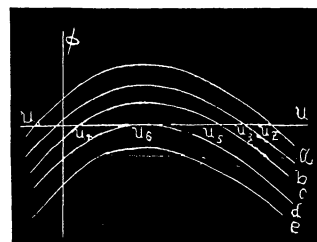


Fig. IV. $a=2$.
Attraction and Repulsion.

(1) In Case III, $a=3$, there are orbits which go from any finite distance to either the origin or infinity, making an infinite number of revolutions around the origin in each case.

(2) In Case IV, $\alpha=2$ (the Newtonian Law), we find for the first time an orbit having two apse distances.

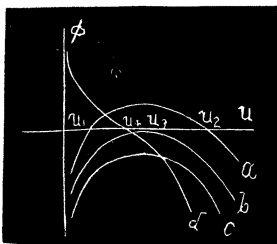


Fig. V. $\alpha=1$.
Attraction and Repulsion.

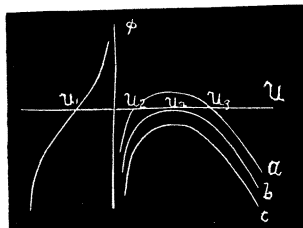


Fig. VI. $\alpha<1$ and even.
Attraction and Repulsion.

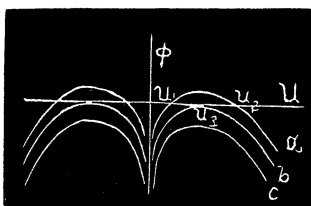


Fig. VII. $\alpha<1$ and odd.
Attraction.

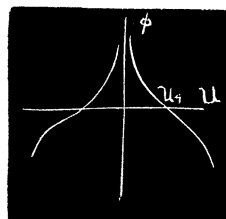


Fig. VII. $\alpha<1$ and odd.
Repulsion.

(3) In Cases V, VI, and VII, $\alpha=2$, the orbits for attraction all have two apse distances, and the orbits lie between the two circles $r=1/u_1$ and $r=1/u_2$, where u_1 and u_2 are the real roots of $\phi(u)=0$. No orbit goes to the origin or to infinity. Periodic orbits may be possible in these cases.

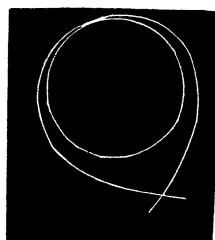


Fig. F.

(4) In Cases I, II, and III, $\alpha>2$, orbits for attraction lie outside of the region between the two circles $r=1/u_1$ and $r=1/u_2$. In Case IV, $\alpha=2$, there are orbits between two circles and also orbits which go to infinity. The Newtonian Law thus gives an intermediate case between the two general classes of orbits.

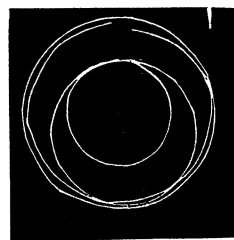


Fig. G.

(5) No orbit in any case has more than two apse distances.

(6) There are orbits which have no apse.

(7) In cases of attraction a circle is a possible orbit for all values of α .

Discussion of T. From equation (7) we get

$$t_1 - t_0 = \int_{u_0}^{u_1} \frac{du}{hu^2 \sqrt{\phi(u)}} = \int_{u_0}^{u_1} \frac{du}{hu^2 \sqrt{\pm \frac{2m^2}{a-1} u^{a-1} - u^2 + c}}, \quad (a \neq 1).$$

It is now shown under what conditions $t_1 - t_0$ is finite, and these criteria are then applied to the discussion of T .

If the limits are finite the integral is finite, if the integrand is infinite of order less than unity at every point of the interval (u_0, u_1) . (1) If the limits do not include $u=0$, the integral is infinite only if $\phi(u)=0$ has a multiple root between or at the limits. (2) If the limits include $u=0$, the integral is infinite unless $\sqrt[\alpha]{\phi(u)}$ is infinite of order greater than one, when it is in general finite. This is equivalent to saying that the integral is finite at $u=0$ only if $\alpha < -1$. (3) If one limit is ∞ and the other so taken that the integrand never becomes infinite between the limits, then the integral is finite if $u^2 \sqrt[\alpha]{\phi(u)}$ at $u=\infty$ is infinite of order greater than unity. In other words the integral is finite if $\alpha > -1$.

Applying these criteria we find that in Cases I, II, III, IV, and V, for both attraction and repulsion, and in Cases VI and VII for attraction, we can make the general statement that all orbits going to infinity and all those which approach a circle asymptotically are not traversed in a finite time, but for all other orbits T is finite. In Cases VI and VII for repulsion we see from criterion (2) that T is infinite if $\alpha \geq -1$, but finite if $\alpha < -1$; hence we have the interesting result that, *if a repulsive force varies directly as a power of the distance greater than unity, the particle goes to infinity in a finite time.*

TABLE OF RESULTS.

Case	Sub-Case	Number of Apses	Variation of r in One Orbit	Θ	Type of Orbit	T
I. $a > 3$ and odd.	(a)	none	$0 \leq r \leq \infty$	finite	Fig. A	∞ at $r = \infty$
	(b)	two	$0 \leq r \leq 1/u_2$ $1/u_1 \leq r \leq \infty$	finite } finite }	Fig. B	finite ∞ at $r = \infty$
	(c)	one	$0 \leq r \leq 1/u_3$ } $1/u_3 \leq r \leq \infty$ }	∞ at $r = 1/u_3$	Fig. C	∞ at $r = 1/u_3$ ∞ at $r = \infty$
			$r = 1/u_3$		circle	
	(d)	one	$r = \infty$			
			$0 \leq r \leq 1/u_4$	finite	Fig. D	finite
	(e)	one	$0 \leq r \leq 1/u_5$	finite	Fig. D	finite
	(f)	one	$1/u_6 \leq r \leq \infty$	finite	Fig. E	∞ at $r = \infty$
	(g)		$r = \infty$			
	(h)		none			
II. $a > 3$ and even.	(a)	one	$0 \leq r \leq \infty$	finite	Fig. A	∞ at $r = \infty$
			$1/u_1 \leq r \leq \infty$	finite	Fig. E	∞ at $r = \infty$
	(b)	two	$1/u_3 \leq r \leq \infty$ $0 \leq r \leq 1/u_3$	∞ } ∞ }	Fig. C	∞ at $r = 1/u_3$ ∞ at $r = \infty$
			$r = 1/u_3$		circle	
			$1/u_2 \leq r \leq \infty$	finite	Fig. E	∞ at $r = \infty$
	(c)	three	$1/u_5 \leq r \leq \infty$ $0 \leq r \leq 1/u_6$	finite } finite }	Fig. B	∞ at $r = \infty$ finite
			$1/u_4 \leq r \leq \infty$	finite	Fig. C	∞ at $r = \infty$
	(d)	two	$r = \infty$			
			$0 \leq r \leq 1/u_7$	finite	Fig. D	finite
	(e)	one	$0 \leq r \leq 1/u_8$	finite	Fig. D	finite

Case	Sub-Case	Number of Apses	Variation of r in One Orbit	Θ	Type of Orbit	T
III. $a=3$.	$m^2 > 1$ { (a)	none	$0 \leq r \leq \infty$	∞ at $r=0$	Fig. A	∞ at $r=\infty$
	(b)	one	$0 \leq r \leq \infty$	∞ at $\begin{cases} r=0 \\ r=\infty \end{cases}$	Fig. A	∞ at $r=\infty$
	(c)	one	$0 \leq r \leq 1/u_1$	∞ at $r=0$	Fig. D	finite
	$m^2 < 1$ { (d)	one	$1/u_2 \leq r \leq \infty$	finite	Fig. F	∞ at $r=\infty$
	(e)	one	$r=\infty$			
	(f)	none	none			
	$m^2 = 1$ { (g)	none	$0 \leq r \leq \infty$	∞ at $r=0$	Fig. A	∞ at $r=\infty$
	(h)	∞	$\begin{cases} r=c \\ 0 \leq c \leq \infty \end{cases}$		circle	
	(i)		none			
For repulsion, see Case I, Repulsion.						
IV. $a=2$.	(a)	two	$1/u_2 \leq r \leq \infty$	finite	Fig. F	∞ at $r=\infty$
			$1/u_1 \leq r \leq \infty$	<i>finite</i>	<i>Fig. E</i>	∞ at $r=\infty$
	(b)	two	$1/u_3 \leq r \leq \infty$	finite	Fig. F	∞ at $r=\infty$
	(c)	two	$1/u_5 \leq r \leq 1/u_4$	finite	Fig. G	finite
	(d)	one	$r=1/u_6$		circle	
	(e)		none			
V. $a=1$.	(a)	two	$1/u_2 \leq r \leq 1/u_1$	finite	Fig. G	finite
	(b)	one	$r=1/u_3$		circle	
	(c)		none			
	(d)	one	$1/u_4 \leq r \leq \infty$	<i>finite</i>	<i>Fig. E</i>	∞ at $r=\infty$
VI. $a < 1$ and even.	(a)	three	$1/u_3 \leq r \leq 1/u_2$	finite	Fig. G	finite
			$1/u_3 \leq r \leq \infty$	<i>finite</i>	<i>Fig. E</i>	$\begin{cases} \text{finite}, a < -1 \\ \infty, a \geq -1 \end{cases}$
	(b)	two	$r=1/u_4$		circle	
			$1/u_1 \leq r \leq \infty$	<i>finite</i>	<i>Fig. E</i>	$\begin{cases} \text{finite}, a < -1 \\ \infty, a \geq -1 \end{cases}$
	(c)	one	$1/u_1 \leq r \leq \infty$	<i>finite</i>	<i>Fig. E</i>	$\begin{cases} \text{finite}, a < -1 \\ \infty, a \geq -1 \end{cases}$
VII. $a < 1$ and odd.	(a)	two	$1/u_2 \leq r \leq 1/u_1$	finite	Fig. G.	finite
	(b)	one	$r=1/u_3$		circle	
	(c)		none			
	(d)	one	$1/u_4 \leq r \leq \infty$	<i>finite</i>	<i>Fig. E</i>	$\begin{cases} \text{finite}, a < -1 \\ \infty, a \geq -1 \end{cases}$